

# Critical Collapse Beyond Spherical Symmetry: General Perturbations of the Roberts Solution

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## Abstract

This paper studies the non-spherical perturbations of the continuously self-similar critical solution of the gravitational collapse of a massless scalar field (the Roberts solution). The exact analysis of the perturbation equations reveals that there are no growing non-spherical perturbation modes.

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## I. INTRODUCTION

Choptuik's discovery of critical phenomena in the gravitational collapse of a scalar field [1] sparked a surge of interest in gravitational collapse just at the threshold of black hole formation. The discovery of critical behavior in several other matter models quickly followed [2–7]. Despite the fact that the evolution equations are very complex and highly non-linear, the dynamics of the near-critical field evolution is relatively simple and, in some important aspects, universal. The critical solution, which depends on the matter model only, serves as an intermediate attractor in the phase space of solutions, and often has an additional peculiar symmetry called self-similarity. The mass of the black hole produced in supercritical evolution scales as a power law

$$M_{\text{BH}}(p) \propto |p - p^*|^\beta, \quad (1)$$

with parameter  $p$  describing initial data, and mass-scaling exponent  $\beta$  is dependent only on the matter model, but not on the initial data family. An interesting consequence of mass scaling which has direct bearing on the cosmic censorship conjecture is the fact that arbitrarily small black holes can be produced in near-critical collapse, with the critical solution exhibiting a curvature singularity and no event horizon.

The explanation of the universality of the critical behavior lies in perturbation analysis and renormalization group ideas [3–5,8]. It turns out that critical solutions generally have only one unstable perturbation mode, making them the most important solutions for understanding the dynamics of field evolution, after the stable ones (flat space and Schwarzschild or Kerr-Newman black hole). As the near-critical field configuration evolves, all its perturbation modes decay, losing information about the initial data and bringing the solution closer to critical, except the one growing mode which will eventually drive the solution to black hole formation or dispersal, depending on its content in the initial data. Thus the critical solution acts as an intermediate attractor (of codimension one) in the phase space of field configurations. Finding the eigenvalue of the growing perturbation mode allows one to calculate important parameters of the critical evolution, the mass-scaling exponent in particular.

An important question is how generic the critical behavior is with respect to initial data, or, in phase space language, how big is the basin of attraction of the critical solution. So far most of the work on critical gravitational collapse, numerical or analytic, has been restricted to the case of spherical symmetry, simply because of the enormous difficulties in treating fully general non-symmetric solutions of Einstein equations. A natural concern is whether the critical phenomena observed so far are limited to spherical symmetry, and whether the evolution of non-spherical data will lead to the same results. The numerical study of Abrahams and Evans on axisymmetric gravitational wave collapse [2] and recent numerical perturbation calculations by Gundlach [9,10] give numerical evidence for the claim that critical phenomena are not restricted to spherical symmetry, and that the critical solutions are indeed attractors in the full phase space. In this paper we search for analytical evidence to support that claim.

One of the few known closed form solutions related to critical phenomena is the Roberts solution, originally constructed as a counterexample to the cosmic censorship conjecture

[11], and later rediscovered in the context of critical gravitational collapse [12,13]. It is a continuously self-similar solution of a spherically symmetric gravitational collapse of a minimally coupled massless scalar field. While it is not a proper critical solution, as it has more than one growing mode [14], it is still a good (and simple) toy model of the critical collapse of the scalar field.

This paper considers fully general perturbations of the Roberts solution in a gauge-invariant formalism. Due to the symmetries of the background, the linear perturbation equations decouple and the variables separate, so an exact analytical treatment is possible. We find that there are no growing perturbation modes apart from spherically symmetric ones described earlier [14]. So all the non-sphericity of the initial data decays in the collapse of the scalar field, and only the spherically symmetric part will play a role in the critical behavior.

To our knowledge, this is the first paper to obtain analytical results on non-spherical critical collapse.

## II. THE ROBERTS SOLUTION

The spacetime we will use as a background in our calculations is a continuously self-similar spherically symmetric solution of the gravitational collapse of a massless scalar field (the Roberts solution). The Einstein-scalar field equations

$$R_{\mu\nu} = 2\phi_{,\mu}\phi_{,\nu}, \quad (2)$$

$$\square\phi = 0 \quad (3)$$

can be solved analytically in spherical symmetry by imposing continuous self-similarity on the solution, i.e. by assuming that there exists a vector field  $\xi$  such that

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu}, \quad \mathcal{L}_\xi \phi = 0, \quad (4)$$

where  $\mathcal{L}$  denotes Lie derivative. Self-similar solutions form a one-parameter family, which is most easily derived in null coordinates [12,13,15]. The critical solution is given by the metric

$$ds^2 = -2 du dv + r^2 d\Omega^2, \quad (5)$$

where

$$r = \sqrt{u^2 - uv}, \quad \phi = \frac{1}{2} \ln \left[ 1 - \frac{v}{u} \right]. \quad (6)$$

The global structure of the critical spacetime is shown in Fig. 1. The influx of the scalar field is turned on at the advanced time  $v = 0$ , so that the spacetime is Minkowskian to the past of this surface. The initial conditions for the field equations (2) and (3) are specified there by the continuity of the solution.

It is instructive to rewrite Roberts solution in new coordinates so that the self-similarity becomes apparent. For this purpose we introduce scaling coordinates

$$x = \frac{1}{2} \ln \left[ 1 - \frac{v}{u} \right], \quad s = -\ln(-u), \quad (7)$$

with the inverse transformation

$$u = -e^{-s}, \quad v = e^{-s}(e^{2x} - 1). \quad (8)$$

The signs are chosen to make the arguments of the logarithm positive in the region of interest ( $v > 0$ ,  $u < 0$ ), where the field evolution occurs. In these coordinates the metric (5) becomes

$$ds^2 = 2e^{2(x-s)} \left[ (1 - e^{-2x})ds^2 - 2dsdx \right] + r^2 d\Omega^2, \quad (9)$$

and the critical solution (6) is simply

$$r = e^{x-s}, \quad \phi = x. \quad (10)$$

Observe that the scalar field  $\phi$  does not depend on the scale variable  $s$  at all, and the only dependence of the metric coefficients on the scale is through the conformal factor  $e^{-2s}$ . This is a direct expression of the geometric requirement (4) in scaling coordinates; the homothetic Killing vector  $\xi$  is simply  $-\frac{\partial}{\partial s}$ .

### III. GAUGE-INVARIANT PERTURBATIONS

To avoid complicated gauge issues of fully general perturbations, we will use the gauge-invariant formalism developed by Gerlach and Sengupta [16,17]. This formalism describes perturbations around a general spherically symmetric background

$$g_{\mu\nu}dx^\mu dx^\nu = g_{AB}dx^A dx^B + r^2\gamma_{ab}dx^a dx^b, \quad (11)$$

which in our case we take to be the Roberts solution (5). Here and later capital Latin indices take values  $\{0, 1\}$ , and lower-case Latin indices run over angular coordinates.  $g_{AB}$  and  $r$  are defined on a spacetime two-manifold, while  $\gamma_{ab}$  is the metric of the unit two-sphere.

Because the background spacetime is spherically symmetric, perturbations around it can be decomposed in spherical harmonics. Scalar spherical harmonics  $Y_{lm}(\theta, \varphi)$  have even parity under spatial inversion, while vector spherical harmonics  $S_{lma}(\theta, \varphi) \equiv \epsilon_a{}^b Y_{lm,b}$  have odd parity. We will only concern ourselves with even-parity perturbations here, since odd-parity perturbations can not couple to scalar field perturbations. We will focus on non-spherical perturbation modes ( $l \geq 1$ ), as the spherically symmetric case ( $l = 0$ ) was studied earlier [14]. For clarity, angular indices  $l, m$  and the summation over all harmonics will be suppressed from now on. The most general even-parity metric perturbation is

$$\begin{aligned} \delta g_{\mu\nu}dx^\mu dx^\nu = & h_{AB}Y dx^A dx^B \\ & + h_A Y_{,b}(dx^A dx^b + dx^b dx^A) \\ & + r^2[KY\gamma_{ab} + GY_{;ab}]dx^a dx^b, \end{aligned} \quad (12)$$

and the scalar field perturbation is

$$\delta\phi = FY. \quad (13)$$

As you can see, metric perturbations are described by a two-tensor  $h_{AB}$ , a two-vector  $h_A$ , and two two-scalars  $K$  and  $G$ ; the scalar field perturbation is described by a two-scalar  $F$ . However, these perturbation amplitudes do not have direct physical meaning, as they change under the (even-parity) gauge transformation induced by the infinitesimal vector field

$$\xi_\mu dx^\mu = \xi_A Y dx^A + \xi Y_{,a} dx^a. \quad (14)$$

One can construct two gauge-invariant quantities from the metric perturbations

$$\begin{aligned} k_{AB} &= h_{AB} - 2p_{(A|B)}, \\ k &= K - 2v^A p_A, \end{aligned} \quad (15)$$

and one from the scalar field perturbation

$$f = F + \phi'^A p_A, \quad (16)$$

where

$$v_A = \frac{r_{,A}}{r}, \quad p_A = h_A - \frac{r^2}{2} G_{,A}. \quad (17)$$

Only gauge-invariant quantities have physical meaning in the perturbation problem. All physics of the problem, including the equations of motion and boundary conditions, should be written in terms of these gauge-invariant quantities. Once gauge-invariant quantities have been identified, one is free to convert between gauge-invariant perturbation amplitudes and their values in whatever gauge choice one desires.

We will work in longitudinal gauge ( $h_A = G = 0$ ), which is particularly convenient since perturbation amplitudes in it are just equal to the corresponding gauge-invariant quantities. The above condition fixes the gauge uniquely for non-spherical modes. (There is some gauge freedom left over for the  $l = 0$  mode, but remember that we are only concerned with higher  $l$  modes.) Expressions for the components of the linear perturbation equations

$$\delta R_{\mu\nu} = 4\phi_{(,\mu}\delta\phi_{,\nu)}, \quad (18)$$

$$\delta(\square\phi) = 0 \quad (19)$$

for a fully general perturbation in longitudinal gauge are collected in Appendix A. By inspection of the  $\theta\varphi$  component of the equations, it is clear that the equations of motion require that  $h_{uv} = 0$  for  $l \geq 1$ . With the change of notation  $h_{uu} = U$  and  $h_{vv} = V$ , the remaining equations of motion for non-spherical modes are

$$4(u^2 - uv)F_{,vu} - uU_{,v} - uK_{,u} + vK_{,v} + vV_{,u} - 2uF_{,u} + 2(2u - v)F_{,v} + 2l(l + 1)F = 0, \quad (20a)$$

$$- 2(u^2 - uv)K_{,uu} + uU_{,u} + (2u - v)(U_{,v} - 2K_{,u}) - 4vF_{,u} + l(l + 1)U = 0, \quad (20b)$$

$$- (u^2 - uv)(U_{,vv} + 2K_{,vu} + V_{,uu}) + uU_{,v} + uK_{,u} - (2u - v)(K_{,v} + V_{,u}) + 2uF_{,u} - 2vF_{,v} = 0, \quad (20c)$$

$$-2(u^2 - uv)K_{,vv} + 2uK_{,v} - uV_{,u} - (2u - v)V_{,v} + 4uF_{,v} + l(l+1)V = 0, \quad (20d)$$

$$2(u^2 - uv)K_{,vu} - uU_{,v} - 2uK_{,u} + (2u - v)(2K_{,v} + V_{,u}) - 2K + l(l+1)K + 2V = 0, \quad (20e)$$

$$(u^2 - uv)(U_{,v} + K_{,u}) + 2vF = 0, \quad (20f)$$

$$(u - v)(V_{,u} + K_{,v}) - 2F = 0. \quad (20g)$$

Equation (20a) comes from the scalar wave equation, and equations (20b–20g) are the  $uu$ ,  $uv$ ,  $vv$ ,  $\theta\theta$ ,  $u\theta$ , and  $v\theta$  components of the Einstein equations, correspondingly. As usual with a scalar field, the system (20) has one redundant equation, so equation (20c) is satisfied automatically by virtue of other equations. Equations (20f) and (20g) are constraints, and the remaining four equations are dynamic equations for four perturbation amplitudes  $U$ ,  $V$ ,  $K$ , and  $F$ .

Boundary conditions for the system (20) are specified at  $v = 0$  and the spatial infinity. Continuity of matching with flat spacetime at the hypersurface  $v = 0$  requires the vanishing of the perturbations there. We also require well-behavedness of the perturbations at  $\mathcal{I}^-$  and  $\mathcal{I}^+$ , so that the perturbation expansion holds. Thus, the boundary conditions are

$$\begin{aligned} U = V = K = F = 0 \text{ at } v = 0, \\ U, V, K, F \text{ are bounded at } u = -\infty \text{ and } v = +\infty. \end{aligned} \quad (21)$$

Equations (20) together with boundary conditions (21) constitute our eigenvalue problem.

#### IV. DECOUPLING OF PERTURBATION EQUATIONS

It is possible to decouple the dynamic equations (20a–20e) by combining them with the constraints (20f) and (20g), and their first derivatives. After somewhat cumbersome algebraic manipulations, which we will not show here, the system of linear perturbation equations (20) can be rewritten as

$$2(u^2 - uv)F_{,vu} - uF_{,u} + (2u - v)F_{,v} + \frac{2vF}{u - v} + l(l+1)F = 0, \quad (22a)$$

$$2(u^2 - uv)U_{,vu} + uU_{,u} + 3(2u - v)U_{,v} + l(l+1)U = 0, \quad (22b)$$

$$2(u^2 - uv)V_{,vu} - 3uV_{,u} - (2u - v)V_{,v} + l(l+1)V = 0, \quad (22c)$$

$$2(u^2 - uv)K_{,vu} - uK_{,u} + (2u - v)K_{,v} - 2K + l(l+1)K = -2V - \frac{4uF}{u - v}, \quad (22d)$$

$$uU_{,v} + uK_{,u} + \frac{2vF}{u - v} = 0, \quad (22e)$$

$$V_{,u} + K_{,v} - \frac{2F}{u-v} = 0. \quad (22f)$$

This decoupled system of partial differential equations can be further simplified by exploiting continuous self-similarity of the background to separate spatial and scale variables. With this intent, we rewrite equations (22) in terms of the scaling coordinates (7)

$$\frac{1}{2} (1 - e^{-2x}) F_{,xx} + F_{,xs} + F_{,s} - 2(1 - e^{-2x}) F + l(l+1) F = 0, \quad (23a)$$

$$\frac{1}{2} (1 - e^{-2x}) U_{,xx} + U_{,xs} - 2U_{,x} - U_{,s} + l(l+1) U = 0, \quad (23b)$$

$$\frac{1}{2} (1 - e^{-2x}) V_{,xx} + V_{,xs} + 2V_{,x} + 3V_{,s} + l(l+1) V = 0, \quad (23c)$$

$$\frac{1}{2} (1 - e^{-2x}) K_{,xx} + K_{,xs} + K_{,s} - 2K + l(l+1) K = -2V - 4e^{-2x} F, \quad (23d)$$

$$U_{,x} - (1 - e^{2x}) K_{,x} + 2e^{2x} K_{,s} - 4(1 - e^{2x}) F = 0, \quad (23e)$$

$$K_{,x} - (1 - e^{2x}) V_{,x} + 2e^{2x} V_{,s} + 4F = 0. \quad (23f)$$

We decompose the perturbation amplitudes into modes that grow exponentially with the scale  $s$  (which amounts to doing Laplace transform on them)

$$\begin{aligned} F(x, s) &= \sum_{\kappa} F_{\kappa}(x) e^{\kappa s}, \\ U(x, s) &= \sum_{\kappa} U_{\kappa}(x) e^{\kappa s}, \\ V(x, s) &= \sum_{\kappa} V_{\kappa}(x) e^{\kappa s}, \\ K(x, s) &= \sum_{\kappa} K_{\kappa}(x) e^{\kappa s}. \end{aligned} \quad (24)$$

The summation runs over the perturbation mode eigenvalues  $\kappa$ , which could, in general, be complex. Modes with  $\text{Re } \kappa > 0$  grow and are relevant for critical behavior, while modes with  $\text{Re } \kappa < 0$  decay and are irrelevant. The growing perturbation mode amplitudes vanish at  $s = -\infty$ , so the boundary condition at  $\mathcal{I}^-$  is satisfied automatically. For clarity, the perturbation mode subscript  $\kappa$  and the explicit summation over all modes will be suppressed from now on, so henceforth  $F$ ,  $U$ ,  $V$ , and  $K$  will mean  $F_{\kappa}$ ,  $U_{\kappa}$ ,  $V_{\kappa}$ , and  $K_{\kappa}$  for the mode with eigenvalue  $\kappa$ .

The decomposition (24) converts the system of partial differential equations (23) into a system of ordinary differential equations, which is much easier to analyze:

$$\frac{1}{2} (1 - e^{-2x}) F'' + \kappa F' + \kappa F - 2(1 - e^{-2x}) F + l(l+1) F = 0, \quad (25a)$$

$$\frac{1}{2}(1 - e^{-2x})U'' + (\kappa - 2)U' - \kappa U + l(l + 1)U = 0, \quad (25b)$$

$$\frac{1}{2}(1 - e^{-2x})V'' + (\kappa + 2)V' + 3\kappa V + l(l + 1)V = 0, \quad (25c)$$

$$\frac{1}{2}(1 - e^{-2x})K'' + \kappa K' + (\kappa - 2)K + l(l + 1)K = -2V - 4e^{-2x}F, \quad (25d)$$

$$U' - (1 - e^{2x})K' + 2\kappa e^{2x}K - 4(1 - e^{2x})F = 0, \quad (25e)$$

$$K' - (1 - e^{2x})V' + 2\kappa e^{2x}V + 4F = 0. \quad (25f)$$

The prime denotes a derivative with respect to spatial variable  $x$ . These equations can be converted into standard algebraic form by the change of variable

$$y = e^{2x}, \quad x = \frac{1}{2} \ln y, \quad (26)$$

so that the system (25) becomes

$$y(1 - y)\ddot{\Phi} + [3 - (\kappa + 3)y]\dot{\Phi} - [3\kappa/2 + l(l + 1)/2]\Phi = 0, \quad (27a)$$

$$y(1 - y)\ddot{U} + [1 - (\kappa - 1)y]\dot{U} - [-\kappa/2 + l(l + 1)/2]U = 0, \quad (27b)$$

$$y(1 - y)\ddot{V} + [1 - (\kappa + 3)y]\dot{V} - [3\kappa/2 + l(l + 1)/2]V = 0, \quad (27c)$$

$$y(1 - y)\ddot{K} + [1 - (\kappa + 1)y]\dot{K} - [\kappa/2 - 1 + l(l + 1)/2]K = 2\Phi + V, \quad (27d)$$

$$\dot{U} + (y - 1)\dot{K} + \kappa K + 2y\Phi - 2\Phi = 0, \quad (27e)$$

$$\dot{K} + (y - 1)\dot{V} + \kappa V + 2\Phi = 0. \quad (27f)$$

The dot denotes a derivative with respect to  $y$ , and we redefined the scalar field perturbation amplitude as  $F = y\Phi$  to cast the equations into standard table form. The boundary conditions (21) are

$$\begin{aligned} U = V = K = \Phi = 0 \text{ at } y = 1, \\ U, V, K, y\Phi \text{ are bounded at } y = +\infty. \end{aligned} \quad (28)$$

Imposed on the decoupled system of ordinary differential equations (27), these boundary conditions give an eigenvalue problem for the perturbation spectrum  $\kappa$ .



## V. PERTURBATION SPECTRUM

In the previous section we formulated an eigenvalue problem for the spectrum of non-spherical perturbations of the critical Roberts solution. We now proceed to solve it. Observe that equations (27a–27d) governing the dynamics of the perturbations are hypergeometric equations of the form

$$y(1-y)\ddot{X} + [c - (a+b+1)y]\dot{X} - abX = 0. \quad (29)$$

Equation (27d) is not homogeneous, but we will deal with that shortly. The hypergeometric equation coefficients are different for equations describing the perturbations  $\Phi$ ,  $U$ ,  $V$ , and  $K$ ; they are summarized in the table below.

	$c$	$a+b$	$ab$
$\Phi$	3	$\kappa + 2$	$\frac{3}{2}\kappa + \frac{1}{2}l(l+1)$
$U$	1	$\kappa - 2$	$-\frac{1}{2}\kappa + \frac{1}{2}l(l+1)$
$V$	1	$\kappa + 2$	$\frac{3}{2}\kappa + \frac{1}{2}l(l+1)$
$K$	1	$\kappa$	$\frac{1}{2}\kappa + \frac{1}{2}l(l+1)$

(30)

Hypergeometric equations have been extensively studied; for complete description of their properties see, for example, [18]. Hypergeometric equation (29) has three singular points at  $y = 0, 1, \infty$ , and its general solution is a linear combination of any two different solutions from the set

$$\begin{aligned} X_1 &= \mathcal{F}(a, b; a+b+1-c; 1-y), \\ X_2 &= (1-y)^{c-a-b} \mathcal{F}(c-a, c-b; c+1-a-b; 1-y), \\ X_3 &= (-y)^{-a} \mathcal{F}(a, a+1-c; a+1-b; y^{-1}), \\ X_4 &= (-y)^{-b} \mathcal{F}(b+1-c, b; b+1-a; y^{-1}), \end{aligned} \quad (31)$$

where  $\mathcal{F}(a, b; c; y)$  is the hypergeometric function, which is regular at  $y = 0$  and has  $\mathcal{F}(a, b; c; 0) = 1$ . Any three of the functions (31) are linearly dependent with constant coefficients. In particular,

$$\begin{aligned} X_2 &= \frac{\Gamma(c+1-a-b)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)} e^{-i\pi(c-b)} X_3 + \\ &\quad \frac{\Gamma(c+1-a-b)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)} e^{-i\pi(c-a)} X_4. \end{aligned} \quad (32)$$

The functions  $X_1$ ,  $X_2$  are appropriate for discussing the behavior of solution near  $y = 1$ , while  $X_3$ ,  $X_4$  give the behavior at infinity

$$\begin{aligned} X_1 &= 1, \quad X_2 = (1-y)^{c-a-b} \text{ near } y = 1, \\ X_3 &= (-y)^{-a}, \quad X_4 = (-y)^{-b} \text{ near } y = \infty. \end{aligned} \quad (33)$$

As we said before, imposing the boundary conditions (28) on solutions of equations (27) leads to a perturbation spectrum. We will now investigate what restrictions the boundary conditions place on the hypergeometric equation coefficients. The vanishing of perturbation amplitudes at  $y = 1$  rules out  $X_1$  as a component of the solution and requires that  $\text{Re}(c - a - b) > 0$  to make  $X_2$  go to zero. The solution  $X_2$  has non-zero content of both  $X_3$  and  $X_4$  by virtue of (32), hence for it to be bounded at infinity, both  $\text{Re } a$  and  $\text{Re } b$  must be positive to guarantee convergence of  $X_3$  and  $X_4$ . So, unless there is degeneracy, the boundary conditions translate to the following conditions on the hypergeometric equation coefficients:

$$\text{Re}(c - a - b) > 0, \quad (34a)$$

$$\text{Re } a, \text{Re } b > 0. \quad (34b)$$

We are now ready to take on system (27). Take a look at equation (27c) for  $V$ . Condition (34a) for it is  $\text{Re } \kappa < -1$ , i.e. there are no growing  $V$  modes! With the amplitude of relevant  $V$  perturbation modes being zero, the constraints (27e) and (27f) become

$$K = -\frac{\dot{U}}{\kappa}, \quad \Phi = \frac{\ddot{U}}{2\kappa}, \quad (35)$$

and right hand side of equation (27d) can be absorbed by the left hand side, making the equation for  $K$  homogeneous (with  $c = 2$ ). Indeed equations (27d) and (27a) for  $K$  and  $\Phi$  are just derivatives of equation (27b) for  $U$

$$y(1-y)\ddot{U} + [1 - (\kappa - 1)y]\dot{U} - [-\kappa/2 + l(l+1)/2]U = 0, \quad (36)$$

which is the homogeneous hypergeometric equation with coefficients

$$c = 1, \quad a + b = \kappa - 2, \quad ab = -\frac{1}{2}\kappa + \frac{1}{2}l(l+1). \quad (37)$$

Imposing the boundary condition at  $y = 1$  for the solution of the above equation and its derivatives, which behave like

$$\left. \begin{aligned} U &\propto (1-y)^{3-\kappa} \\ K &\propto (1-y)^{2-\kappa} \\ \Phi &\propto (1-y)^{1-\kappa} \end{aligned} \right\} \text{ near } y = 1, \quad (38)$$

produces restriction on the non-spherical mode eigenvalue

$$\text{Re } \kappa < 1, \quad (39)$$

which is the strongest of restrictions (34a) for equations for  $U$ ,  $K$ , and  $\Phi$ . But then

$$\text{Re } a + \text{Re } b = \text{Re } \kappa - 2 < -1, \quad (40)$$

and hence  $\text{Re } a$  and  $\text{Re } b$  can not be both positive, and so the boundary condition at infinity can not be satisfied. A more careful investigation of degenerate cases of relation (32) shows that the contradiction between boundary conditions at  $y = 1$  and infinity still persists if  $V = 0$ . It can only be resolved by the trivial solution  $U = K = \Phi = 0$ . Thus we have shown

that there are no growing non-spherical perturbation modes around the critical Roberts solution.

In fact, an even stronger statement is true. The contradiction between boundary conditions at  $y = 1$  and infinity can not be resolved by a non-trivial solution so long as  $V = 0$ , i.e. so long as  $\text{Re } \kappa \geq -1$ . Hence non-trivial non-spherical perturbation modes of critical Roberts solution must decay faster than  $e^{-s}$ .

## VI. CONCLUSION

In this paper we used the gauge-invariant perturbation formalism to explore the critical behavior in the gravitational collapse of a massless scalar field. Perturbing around a continuously self-similar critical solution (the Roberts solution), we obtained an eigenvalue problem for the spectrum of perturbations. The remarkable feature of this model of critical scalar field collapse is that it allows an exact analytical treatment of the perturbations as well as of the critical solution, due to the highly symmetric background.

An exact analysis of the perturbation eigenvalue problem reveals that there are no growing non-spherical perturbation modes. However, there are growing spherical perturbation modes. Their spectrum is continuous and occupies a big chunk of the complex plane [14]. In view of these findings, the following picture of dynamics of scalar field evolution near self-similarity emerges: As we evolve generic initial data which is sufficiently close to the critical Roberts solution, non-spherical modes decay and the solution approaches the spherically symmetric one. Asymmetry of the initial data does not play a role in the collapse. The growing spherical modes, on the other hand, drive the solution farther away from the continuously self-similar one. In this sense, the critical Roberts solution is an intermediate attractor for non-spherical initial data.

An interesting question, which is not answered by perturbative calculations, is the further fate of the scalar field evolution as it gets away from the Roberts solution. In all likelihood, it evolves towards the discretely self-similar Choptuik solution, which is a local attractor of lower codimension (one), as the continuous self-similarity is broken by oscillatory growing modes. After staying near the Choptuik solution for a while, the scalar field will eventually disperse or settle into a black hole, with these final states being global attractors in the phase space of field configurations. This evolution from attractor to attractor in phase space is somewhat analogous to a ball rolling down the stairs, going from a step to a lower step, until it reaches the bottom.

The results of this paper shed some light at the complicated problem of critical collapse of generic initial data from the analytical viewpoint, confirming the hypothesis that critical phenomena are not restricted to spherical symmetry. Investigation of the fate of a scalar field solution as it breaks away from continuous self-similarity, as outlined above, will further our understanding of the dynamics of scalar field collapse, and presents an interesting (and challenging) analytical problem. Numerical simulations might also help to establish a clearer picture of near-critical scalar field evolution.

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## APPENDIX A: PERTURBATIONS IN LONGITUDINAL GAUGE

In this appendix we collect expressions for components of the perturbed Einstein-scalar equations calculated in longitudinal gauge ( $h_A = G = 0$ ). The perturbed metric in longitudinal gauge is

$$ds^2 = h_{uu}Y du^2 - 2(1 - h_{uv}Y)du dv + h_{vv}Y dv^2 + (1 + KY)r^2 d\Omega^2, \quad (\text{A1})$$

and the perturbed scalar field is

$$\phi = \frac{1}{2} \left[ 1 - \frac{v}{u} \right] + FY. \quad (\text{A2})$$

The Einstein equations for scalar field are equivalent to the vanishing of the tensor  $E_{\mu\nu} = R_{\mu\nu} - 2\phi_{,\mu}\phi_{,\nu}$ . Its non-trivial components, calculated to the first order in the perturbation amplitude using the above metric and scalar field, are

$$\begin{aligned} E_{uu} = \frac{1}{2} \Big[ & -2(u^2 - uv)K_{,uu} + uh_{uu,u} \\ & + (2u - v)(h_{uu,v} - 2h_{uv,u} - 2K_{,u}) \\ & - 4vF_{,u} + l(l+1)h_{uu} \Big] \frac{Y}{u^2 - uv}, \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} E_{uv} = -\frac{1}{2} \Big[ & (u^2 - uv)(h_{uu,vv} - 2h_{uv,vu} + h_{vv,uu} + 2K_{,vu}) \\ & - uh_{uu,v} + (2u - v)(h_{vv,u} + K_{,v}) - uK_{,u} \\ & + 2vF_{,v} - 2uF_{,u} - l(l+1)h_{uv} \Big] \frac{Y}{u^2 - uv}, \end{aligned} \quad (\text{A3b})$$

$$\begin{aligned} E_{vv} = \frac{1}{2} \Big[ & -2(u^2 - uv)K_{,vv} + 2uh_{uv,v} \\ & - uh_{vv,u} - (2u - v)h_{vv,v} + 2uK_{,v} \\ & + 4uF_{,v} + l(l+1)h_{vv} \Big] \frac{Y}{u^2 - uv}, \end{aligned} \quad (\text{A3c})$$

$$\begin{aligned} E_{u\theta} = -\frac{1}{2} \Big[ & (u^2 - uv)(h_{uu,v} - h_{uv,u} + K_{,u}) \\ & + (2u - v)h_{uv} + 2vF \Big] \frac{Y_{,\theta}}{u^2 - uv}, \end{aligned} \quad (\text{A3d})$$

$$E_{u\varphi} = -\frac{1}{2} \left[ (u^2 - uv)(h_{uu,v} - h_{uv,u} + K_{,u}) + (2u - v)h_{uv} + 2vF \right] \frac{Y_{,\varphi}}{u^2 - uv}, \quad (\text{A3e})$$

$$E_{v\theta} = -\frac{1}{2} \left[ (u - v)(-h_{uv,v} + h_{vv,u} + K_{,v}) - h_{uv} - 2F \right] \frac{Y_{,\theta}}{u - v}, \quad (\text{A3f})$$

$$E_{v\varphi} = -\frac{1}{2} \left[ (u - v)(-h_{uv,v} + h_{vv,u} + K_{,v}) - h_{uv} - 2F \right] \frac{Y_{,\varphi}}{u - v}, \quad (\text{A3g})$$

$$E_{\theta\theta} = \frac{1}{2} \left[ 2(u^2 - uv)K_{,vu} - uh_{uu,v} + (2u - v)(h_{vv,u} + 2K_{,v}) - 2uK_{,u} - 2h_{uv} + 2h_{vv} - 2K + l(l + 1)K \right] Y + h_{uv}Y_{,\theta\theta}, \quad (\text{A3h})$$

$$E_{\varphi\varphi} = \sin^2 \theta E_{\theta\theta}, \quad (\text{A3i})$$

$$E_{\theta\varphi} = h_{uv}(Y_{,\theta\varphi} - \cot \theta Y_{,\varphi}). \quad (\text{A3j})$$

The scalar field equation requires the vanishing of  $\square\phi$ , which, calculated to first order using the above metric and scalar field, is

$$\square\phi = \frac{1}{2} \left[ -4(u^2 - uv)F_{,vu} + uh_{uu,v} - vh_{vv,u} + uK_{,u} - vK_{,v} + 2uF_{,u} - 2(2u - v)F_{,v} - 2l(l + 1)F \right] \frac{Y}{u^2 - uv}. \quad (\text{A4})$$

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# FIGURES

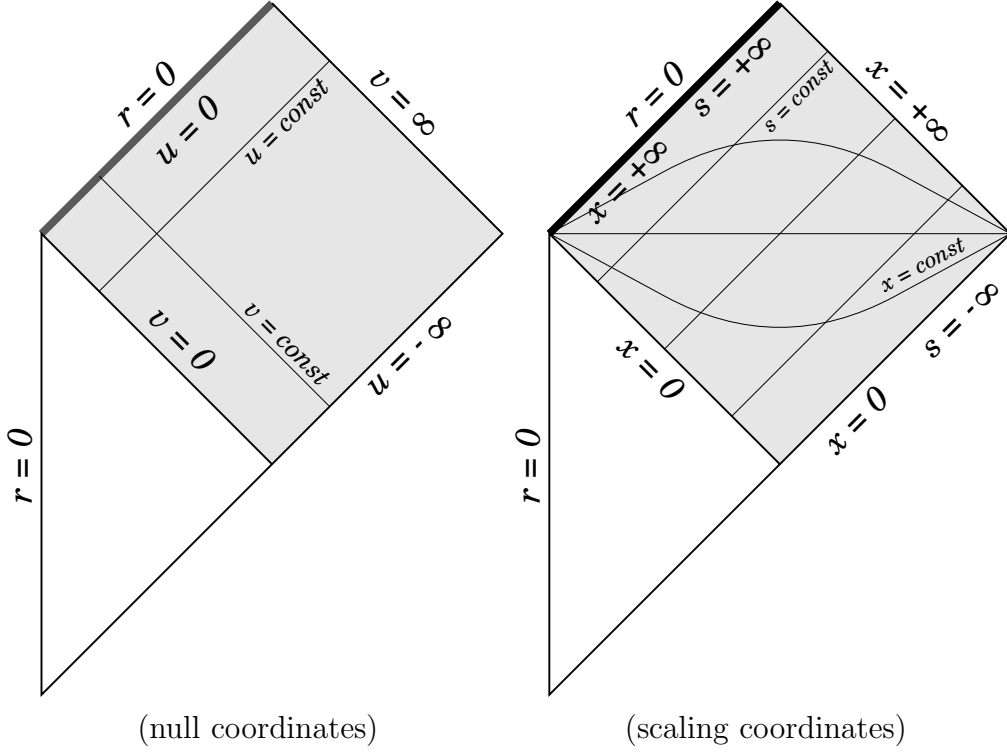


FIG. 1. Global structure of the Roberts solution: The scalar field influx is turned on at  $v = 0$ ; spacetime is flat before that. Field evolution occurs in the shaded region of the diagram, and there is a null singularity in the center of the spacetime.